UNIVERSITY OF ILLINOIS ENGINEERING EXPERIMENT STATION CIRCULAR NO. 79

A Turbulence Primer

by

James M. Robertson

UNIVERSITY OF ILLINOIS BULLETIN

Edited by
James M. Robertson
Michael B. Shelly

Price: $1.00

UNIVERSITY OF ILLINOIS BULLETIN

Volume 62, Number 71; March, 1965. Published nine times each month by the University of Illinois. Entered as second-class matter December 11, 1912, at the post office at Urbana, Illinois, under the Act of August 24, 1912. Office of Publication, 114 Altgeld Hall, Urbana, Illinois 61803.

A TURBULENCE PRIMER

An Introduction to the Basic Phenomena,
Modes of Characterization, and
Terminology of Turbulence

by James M. Robertson
Professor of Theoretical and Applied Mechanics

ENGINEERING EXPERIMENT STATION CIRCULAR NO. 79

SUMMARY

Turbulence is defined as a random motion which occurs in fluid flows. The randomness is of such nature that the velocity at one instant is still correlated with that in the next and in decreasing amounts at succeeding instants. An important characteristic is that the energy in turbulence is distributed over a hierarchy of sizes. A distinction is thus made between turbulence and vortex motions, which do not have these characteristics although they tend to break down to form turbulence. The technical meanings of intensity and scale for characterizing turbulence are identified. The consideration of the production, convection, and diffusion, and dissipation (transfer to heat) of the turbulence energy by various fluid dynamic actions is introduced and some information is presented on the character of turbulence in a few typical flows (behind grids, in boundary layers and pipe flows, and in jets). Use of the momentum-transfer mixing-length model used to acquire technological solutions to the mean flow problems of turbulent motions is presented with an indication of the results obtained.

This report is an introduction to turbulence for engineers and researchers in technical fields where turbulent motions are encountered. Many of those for whom this is written have had little contact with advanced fluid mechanics and little opportunity to delve far into the mysteries of turbulence. And yet they need some understanding of our modern scientific concepts of the nature of turbulence as well as an introduction to the terminology conventionally employed in this field. Some basic knowledge of fluid mechanics is presupposed, however. Since this report was prepared, a more erudite discussion with an indication of current research avenues in turbulence has been presented by S. Corrsin ("Turbulent Flow," The American Scientist, Vol. 49, pp. 300--323). Additional general references on turbulence are listed at the end of this discussion.

ACKNOWLEDGEMENT

This report is a reprint, with slight changes and some additions, of a report prepared for Research and Development Division of Consolidated Water Power and Paper Company, Wisconsin Rapids, Wisconsin.

CONTENTS

FIGURES

TABLES

I. DEFINITION OF TURBULENCE

The ordinary dictionary definition of turbulence as a state or condition of being violently disturbed is much too inclusive for use in technical considerations of this more common type (as opposed to laminar-viscous) of fluid motion. Perhaps the most generally accepted technical definition is one which Th. Von Karman* quoted from G. I. Taylor in 1937. Where two such eminent authorities in the field agree, one can but follow. "Turbulence is an irregular motion which in general makes its appearance in fluids, gaseous or liquid, when they flow past solid surfaces or even when neighboring streams of the same fluid flow past or over one another." As Karman noted "the emphasis in this definition lies in the word irregular." Taylor said further "the actual motion is usually so irregular that very little is known of its details." More recently, in 1953, another expert, G. K. Batchelor pointed out** that the distinguishing feature of turbulence in a flow is that "the velocity at any given time and position is not found to be the same when it is measured several times under seemingly identical conditions . . . . the velocity takes random values which are not determined by the ostensible, or controllable, or "macroscopic", data of the flow, although we believe that the average properties of the motion are determined uniquely by the data . . . ."

An essential feature of turbulence is the fact that the motion is random. It is continuous, however, in the sense that for very short time intervals there is a distinct correlation between velocity measured at a point and that measured at the same point a short time interval later or before. This feature is apparent in the velocity versus time plot of Figure 1. The lack of periodicity or the randomness is essential. Thus, vortex motions are not included in the technical definition of turbulence, although they are often the origin of turbulence. The wake behind a body moving in a fluid may initially consist of a separated or discontinuous flow region. The surfaces of separation roll up into vortices, which appear as vortex streets. Alternatively the Karman vortex street appears to be formed right behind the body. It is only after the flow has progressed for some distance from the body that the vortices decompose slowly into the random motion of turbulence. This is in spite of the fact that, qualitatively, turbulence is described as a hierarchy of eddies moving along with the mean motion.

* Journal Royal Aeronautical Society, December 1937.

** The Theory of Homogeneous Turbulence, Cambridge University Press, 1953, p. 1.

Another feature of turbulence is the hierarchy of scales, loosely termed eddies -- with a transfer of kinetic energy down the scales from the larger to the smaller. The energy is dissipated, or transferred into heat via the action of viscosity at the smallest scales, i.e. by the smallest eddies. The energy is extracted from the mean flow by the large scales or eddies so as to maintain the overall turbulence structure constant. To quote a ditty attributed to L. F. Richardson and well known in the turbulence field,

> "Big whirls have little whirls
> Which feed upon their velocity
> Little whirls have smaller whirls
> And so on to viscosity."

The term whirl is used as synonymous with eddy, meaning (in re turbulence) an irregular type of rolling motion which would be termed a vortex if it were not random in time and space. No precise definition of an eddy is possible since it is merely a concept -- albeit a most useful one.

Turbulent flows may be classified according to their homogeniety and isotropy as well as by the manner in which they occur. Even though some of the relevant terms have not, as yet, been defined here, it seems appropriate to introduce at this point such classifications of turbulent flows. Turbulence is defined as homogeneous when its scale and intensity are independent of coordinate position. It is further defined as being isotropic when these characteristics are independent of direction -- thus the intensity of the turbulent fluctuations are the same in all directions. The various classes of turbulent flows are indicated in Table 1. Following considerations introduced in a later section, it turns out that there are quite significant differences between the turbulence production and dissipation in the several flow classifications. As indicated in the table, homogeneous isotropic turbulence is approximately realized in at least two kinds of flows. The more intensively studied of these is that of air tunnel turbulence. Much of the development of the statistical theory of turbulence stems from the study of the turbulence found behind grids or screens in air tunnels and affecting the drag of models tested therein. This turbulence is only approximately homogeneous since it slowly decays with distance from the producing grid.

Turbulence resulting from the action of turbulent shear on mean velocity gradients is much more common -- and more complicated. The simplest shear flow turbulence is produced in plane Couette flow and was dubbed "homologous" by von Karman in 1937, since the turbulence is homogeneous when the shear and velocity gradients are constant. The present author has had some success in pro-

Table 1

Turbulent Flow Classifications

	Occurrence: Type of Flow	Energy Considerations: Production	Transverse Motion	Dissipation
A. Homogeneous 1. Isotropic	a. air tunnel (decaying)	0	0	0 (slow decay)
	b. wide open channel* (two-dimensional aspects**)	0	0	0
	c. approx. in center region of conduits	0	0	0
2. Homologous	a. plane Couette flow*** a constant shear flow, as by one plane shearing parallel to another	pr. =dis.	0	pr. =dis.
B. Nonhomogeneous (and anisotropic)	General Shear Flows (shear varies)			
	a. fully developed pipe flow and flat plate boundary layers (invariant in flow direction)	production	convection and diffusion	dissipation
	b. most general case -- adverse pressure gradient flows	production	convection and diffusion	dissipation

* in plane; production, etc., occurs normal thereto.

** see G. T. Orlob in Proceedings ASCE, Journal Hydraulics Division, September 1959, p. 75.

*** see Th. von Karman in Journal Aeronautical Sciences, Vol. 4 (1937), p. 131.

ducing homologous turbulence†. More common shear flows, as in pipes or boundary layers are strongly anisotropic and nonhomogeneous. In fully developed flows (i.e. flow pattern invariant in flow direction) the turbulence, as is the mean flow, is a function only of the lateral space coordinate. In adverse pressure gradient flows, the turbulence and mean-motion characteristics vary in the flow direction as well.

Turbulence may be measured indirectly, as via diffusion studies (cf. section IX), or directly via fast-response velocity instrumentation. The common instrument is the hot-wire anemometer; specific reference to experimental techniques in this presentation assumes the use of that instrument.

II. REPRESENTATION OF TURBULENCE

Discussion of turbulence does not progress very far before the fluctuations must be characterized and certain representative terms come into use. These will be briefly defined at this point. Due to the random nature of the turbulent motions, analysis and solution of problems must be in terms of the statistical approach. Such an approach was inaugurated by Osborne Reynolds in 1894, when he assumed that the flow could be divided into temporal-mean and turbulent parts, thus the velocity component in the mean-flow or x direction is written:

$$u = \bar{u} + \underset{\sim}{u}$$

where $\bar{u}$ = mean velocity part, averaged over a relatively long time period

$\underset{\sim}{u}$ = turbulent velocity fluctuation part

Similarly, in the y direction (transverse to the mean flow) and z direction (normal to the xy plane)

$$v = \bar{v} + \underset{\sim}{v} \qquad \text{and} \qquad w = \bar{w} + \underset{\sim}{w}$$

The fluctuations $\underset{\sim}{u}$, $\underset{\sim}{v}$, and $\underset{\sim}{w}$ are treated and

† cf. Proceedings 6th Midwest Conference on Fluid Mechanics, 1959.

characterized by statistical techniques.

The basic equations of turbulent motion were derived by Reynolds with the aid of this division into turbulent and mean motions. The above expressions for the velocity components were substituted into the general equations of viscous-fluid motion (Navier-Stokes equations) and the mean or temporal average value taken. With the aid of certain averaging rules, the equations could be written in a form similar to the original Navier-Stokes equations with but two differences. The velocities all appeared as mean values, $\bar{u}$ etc., and the gradients of certain new terms appeared. These latter terms appear as additional stresses and are now termed the Reynolds stresses. The Reynolds stresses can be written in tensor form*, thus:

$$T'_{ij} = -\rho\overline{\underset{\sim}{u}_i \underset{\sim}{u}_j} = \rho\overline{u_i u_j}$$

$$= \begin{bmatrix} \tau'_{xx} = \sigma'_x & \tau'_{xy} & \tau'_{xz} \\ \tau'_{yx} & \tau'_{yy} = \sigma'_y & \tau'_{yz} \\ \tau'_{zx} & \tau'_{zy} & \tau'_{zz} = \sigma'_z \end{bmatrix}$$

$$= -\rho \begin{bmatrix} \overline{u^2} & \overline{uv} & \overline{uw} \\ \overline{vu} & \overline{v^2} & \overline{vw} \\ \overline{wu} & \overline{wv} & \overline{w^2} \end{bmatrix}$$

where the bar over the product terms means that a temporal average is taken and the wavy underline indicating the fluctuating quantity is omitted, as being evident from the connotation**. The symbol ρ represents the fluid mass per unit volume (lb. - force sec^2 per ft.^{4}). This is a symmetrical stress tensor -- as $\overline{vu} = \overline{uv}$ etc. -- so that there are only six unknown stresses, but the Reynolds equations are still quite intractible. Whereas, in viscous-laminar motions, it is most difficult to find solutions for three Navier-Stokes equations together with the continuity equation when three velocity components and a pressure are unknowns, in the turbulent case there are still only four equations with the same four unknowns plus the additional six turbulence unknowns. It is no wonder that turbulent flow problems are so difficult of solution.

* For the present purposes, this form merely represents a compact way of indicating the nine terms.

** Actually with $\bar{v} = 0 = \bar{w}$ application of the averaging rules indicates that $\overline{uv} = \overline{\underset{\sim}{u}\underset{\sim}{v}}$, etc. Thus, $\overline{uv} = \overline{(\bar{u} + \underset{\sim}{u})(\bar{v} + \underset{\sim}{v})} = \overline{(\bar{u} + \underset{\sim}{u})\,\underset{\sim}{v}} = \overline{\bar{u}\underset{\sim}{v}} + \overline{\underset{\sim}{u}\underset{\sim}{v}} = \overline{\underset{\sim}{u}\underset{\sim}{v}}$ as $\overline{\bar{u}\underset{\sim}{v}} = \bar{u}\bar{\underset{\sim}{v}}$ and $\bar{\underset{\sim}{v}} = 0$ by definition of $\underset{\sim}{v}$.

Many fluid flow problems are two dimensional*, or approximately so, and this permits a simplification in expression, which is useful in considering turbulence. The presentation and discussion herein will be for two dimensional turbulent flows as this is complicated enough. In wide rectangular conduits or most boundary layer flows, the motion may be considered plane two dimensional, with the general motion in the x direction and the variation in velocity quantities occurring mainly in the y direction normal to the planes between which, or along which, the flow occurs. Pipe flows (i.e., in circular conduits) are axisymmetric two dimensional and are completely described in meridional planes passing through the pipe axis. The primary independent geometric variable is that in the radial direction, either as the radius r or as y, the distance from the conduit wall. The Reynolds stress tensor in plane two dimensional flows is simplified to three terms, i.e.

$$T'_{ij} = \begin{bmatrix} \sigma'_x & \tau'_{xy} \\ \tau'_{xy} & \sigma'_y \end{bmatrix} = - \begin{bmatrix} \rho\overline{u^2} & \rho\overline{uv} \\ \rho\overline{uv} & \rho\overline{v^2} \end{bmatrix} \qquad (1)$$

The cross-product term $-\rho\overline{uv}$ is recognized as the turbulent shear stress, or merely the shear stress since, except in localized regions, the laminar-viscous shear stress in turbulent flows is negligible. The other two (normal stress) terms play a less clearly understandable role as stresses, but are nonetheless of great value in describing and interpreting turbulence since they represent measures of the intensity level of the turbulence.

Returning to the concept of turbulence as a random phenomenon and the use of statistical means of representation, one is led to considerations of the distributions of the velocity fluctuation magnitudes over the range of possible values. The probability (or amplitude) density function $P(\underset{\sim}{u})$ is used to indicate the distribution. This function is an indication of the probability that a given instantaneous velocity will lie within an increment δu of a specific value u. Experimental determinations of this function indicate that it is usually close to the Gaussian or normal error distribution, as indicated in Figure 1. With the ordinate adjusted so that the total area under the curve is unity, the normal error distribution has the equation

* A two-dimensional motion is one whose characteristics are completely described in terms of two coordinates. Thus the flow of a fluid about or past a cylinder, such as a smoke stack, may be depicted on a plane normal to the generators of the cylinder, except near the ends of the cylinder (top and bottom of stack) where three-dimensional effects appear. In the mid-region the flow patterns for all planes appear the same.

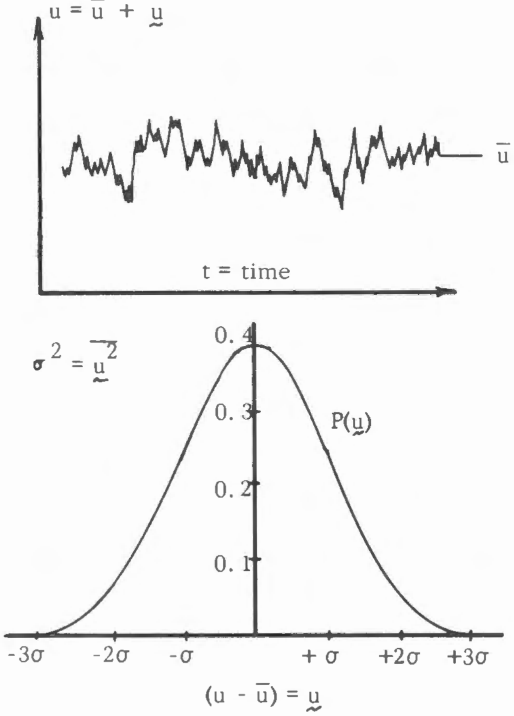

Figure 1. Turbulent Velocity

$$P(\underset{\sim}{u}) = \frac{1}{\sigma\sqrt{2}}\, e^{-(u - \bar{u})^2/2\sigma^2} \qquad (2)$$

where $\sigma^2 = \overline{\underset{\sim}{u}^2}$ is termed the variance and its square root $\sqrt{\overline{\underset{\sim}{u}^2}} = u'$ the standard deviation of the distribution. The probability density function as shown in Figure 1 is a hat-shaped curve symmetrical about $\bar{u}$ where it attains a value of very close to 0.40. The standard deviation is a measure of the breadth of the curve: $P(\underset{\sim}{u})$ is reduced to half the peak value for $\underset{\sim}{u} = u - \bar{u}$ values of 1.18σ and to one percent of the peak value at 3.04σ. Thus, instantaneous velocity fluctuations greater than three times the standard deviation are not likely. The symbol u' is commonly employed for the root-mean-square value of the turbulent velocity fluctuation. As apparent from the discussion in the preceding paragraph, this is also the square root of the magnitude of a normal Reynolds stress divided by the fluid density. A similar concept and definition is used for v' and w' as related to the other normal Reynolds stresses.

Since the r.m.s. value of the turbulence velocity fluctuation is definite quantity in a given turbulent motion and represents a suitable statistical measure of the magnitude of the fluctuations, it is termed the intensity or level of the turbulence. Sometimes the ratio of this value to the mean flow velocity (i.e. the relative intensity), is termed the intensity. Also when the turbulence is not isotropic the mean value of the three components may be used as more appropriate. Thus:

Turbulence intensity is u' or $u'/\bar{u}$

$$\text{or } 1/\bar{u}\ \frac{(u'^2 + v'^2 + w'^2)}{3}$$

Which one of these latter two is used depends to a great extent upon how many of the turbulent velocity components have been measured.

When the nature of turbulent motion in a plane two-dimensional flow is considered, the instantaneous velocity vector q is seen to involve a mean and two turbulence terms,*

$$q = \bar{q} + \underset{\sim}{q} \quad = \quad \bar{u} + \underset{\sim}{u} + \underset{\sim}{v}.$$

This situation is depicted in Figure 2.

Each one of the fluctuating velocity components satisfies a probability density function, $P(\underset{\sim}{u})$ or $P(\underset{\sim}{v})$ as shown. Only in isotropic turbulence are the standard deviations u' and v', indicating the spread of these two functions, the same. The velocity variation is random (although continuous) so that a short time later the instantaneous velocity will have quite a different value than the particular one indicated. However, within the one percent limits noted on the P function, q will always occur in the dotted region indicated.

As implied by the elongated trend of the dotted instantaneous velocity regions, the velocity fluctuation components are generally related or, to use the statistical term, correlated. As will be demonstrated in the discussion of the mixing-length model of turbulent motion, the correlation is negative when a velocity gradient $(\bar{u} = \bar{u}(y))$ occurs so that a negative $\underset{\sim}{u}$ tends to appear with a positive $\underset{\sim}{v}$. If the correlation were perfect the dotted region would be a straight line, of slope given by $-v'/u'$, but it is not, as indicated by Figure 2. However, the existence of correlation is evidence of a turbulent shear stress, while the dashed region is a circle (zero correlation) when there is no shear and velocity gradient as in isotropic turbulent motions. In statistics the correlation between quantities is indicated in terms of a correlation coefficient, defined as the mean product of the two variants divided by the product of their standard deviations. In the case of u and v the correlation coefficient is

* The x direction is conventionally taken as the general mean-flow direction, so that $\bar{v} = 0$. Although $w' \neq 0$ even in plane motions, it is ignored in this discussion, in the interest of clarity

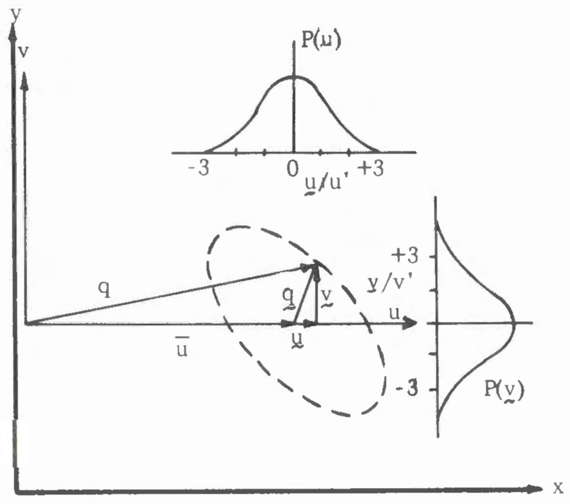

Figure 2. Composition of Instantaneous Velocity.

$$K = \frac{\overline{uv}}{u'v'}$$

This is termed the shear correlation coefficient, since $-\rho\overline{uv}$ is the turbulent shear. In many flows this has a value of about one-half.

III. CORRELATION, SCALE, AND SPECTRUM

The correlation coefficient just defined is only one of several utilized in describing turbulence. Other correlation coefficients are needed to obtain definite measures of the scale of the turbulence. In fact, turbulence scale cannot be defined quantitatively except through correlation coefficients. It is first necessary to consider just what is meant by scale. Qualitatively this implies the size of the eddies making up the turbulence. Although these vary in size over a considerable range, one can conceive of some typical or average size. Actually two specific scales will be defined, one representing the size of the larger eddies and the other the size of the smaller eddies. These are known as the macro, or integral scale, and the micro scale.

In considering a turbulent motion and its effect on an object or structure, it is most important to compare the scale of the turbulence to the size of the body. This is apparent if one considers say, a circular cylinder in an air stream. This cylinder might be the 6-foot diameter stack on a power plant or the 1/4 inch diameter radio antenna on a car. Eddies of several inches to several feet in scale will appear as turbulence to the stack but merely as slow variations in speed to the antenna. The motion of the air in the atmosphere has various scales depending upon the problems considered. Consider for the moment, time rather than length scales. If a velocity indicating instrument such as an anemometer is placed on the top of a building or tower to obtain indications of the turbulence in the atmosphere, different results are obtainable depending upon the length of observation period. If a one-minute observation period is employed, a turbulence intensity can be evaluated and an apparent mean velocity may be identified. If the measurements extend over an hour's time, a definite gustiness of the flow appears and the one-minute mean velocity obtained is found to vary; in fact it may be taken as a turbulent velocity and a new mean velocity defined in terms of the 60 readings (actually more readings are needed) taken at one-minute intervals. This process continues as the time interval is lengthened. The wind speed varies over the period of the day, month, season, year, and the definition of turbulence and mean velocity is found to depend upon the period of interest and response of the instrument used to measure it. This situation is obviously relevant to questions of geometric as well as temporal scale, as noted in connection with the stack and the antenna. In general, the relevant geometric and temporal scales are obvious. Thus for a pipe flow, one is not usually interested in a period of time so long that it includes starting up and stopping the flow through the pipe, rather a period of time is taken during which the rate of flow through the pipe is constant.

Since it is not possible to look at a turbulent flow and see the various-sized eddies present* (if they really exist), some more specific means of determining turbulence scale is necessary. This is accomplished with the aid of velocity correlation measurements. Besides the shear correlation defined in the previous section, many kinds of correlation are considered in the analysis of turbulence -- double (involving two velocities), triple (three velocities), and multiple correlations, as well as those between velocity and pressure. An approach to solving the general problems of turbulence is obtained through considerations of the relations between these correlation coefficients. In this presentation only the simpler double correlations will be introduced.

Fluid motions are usually considered via the Eulerian approach to kinematics in which the occurrences at a given point are noted as fluid streams past. Sometimes, particularly in considerations of turbulent diffusion, it is more appropriate to utilize the Lagrangian approach which considers the dynamics of a fluid particle as it moves through the fluid. A Lagrangian double velocity correlation coefficient, in terms of the x component of velocity, is then defined as

---

* Instantaneous photographs of turbulent flow, taken both with camera fixed and moving with the fluid do seem to show definite eddies, but these are still difficult of quantitative analysis, in terms of defining specific sizes.

$$R_{L_t} = \frac{\overline{u_t u_{t+\delta t}}}{u'^2} \qquad (3)$$

where u_t is the velocity fluctuation of a particle at some instant and $u_{t+\delta t}$ that of the same particle at time δt later. It is assumed that the turbulence is homogeneous, i.e., u' is independent of location, at least for the short time intervals involved. If the velocity versus time plot of Figure 1 represents a Lagrangian measurement, it is apparent by inspection that for very short time intervals the correlation coefficient must be unity, while for larger times it quickly drops to zero, i.e. R is a function of δt. The variation is approximately as $e^{-\delta t}$. The correlation just defined might be termed an autocorrelation, however more commonly the autocorrelation coefficient is defined in the Eulerian fashion, by observing sequential values of the velocity at a given point as time progresses. The above relation for R is still employed except that the subscript L (for Lagrangian) is omitted and the velocities are measured at a given point as time progresses. Where necessary to avoid confusion with R_L the more conventional Eulerian may be indicated as R_E, but in general the subscript will be omitted. The Eulerian autocorrelation coefficient R_t is relatively easily found by obtaining a time record of the velocity component fluctuation at some point, as indicated in Figure 1, and then picking off the velocities. This is a rather laborious procedure and, although electronic means using delay circuits are possible, the autocorrelation is not often evaluated.

A more common type of Eulerian correlation coefficient measured in turbulence studies is that taken between two velocity components at different locations. Thus, $R_{u,x}$ represents the correlation coefficient taken for the u velocity component at two different locations separated by distance x, or

$$R_{u,x} = \frac{\overline{u_x u_{x+\delta x}}}{u'^2} \, . \qquad (4)$$

It is again assumed that the r.m.s. value u' is independent of x at least for the short distances of interest. The definitions for $R_{u,y}$, $R_{v,x}$, etc. are similar. As one might expect, these correlation coefficients are functions of the spacing δx or δy at which they are measured. Figure 3 indicates the mode of variation found. At moderate distances the variation tends to follow the negative exponential noted previously. Measurement of these spatial, or Eulerian, double correlations is commonly achieved with a hot-wire anemometer using two probes which are separated by the desired distance. Actual calculation of the mean product is accomplished electronically.

Inspection of correlation curves in terms of the concept of turbulence scale suggests that their extent does form a definite indication of scale. However, as apparent from Figure 3, it is not adequate to pick off the point where R reaches zero as a measure of the maximum eddy size. Although this is indeed what is indicated, the curves tend to approach zero asymptotically so that the definition is vague. Instead an <u>integral scale</u>, indicating the mean eddy size, is <u>defined as the area</u> under a correlation curve, thus

$$L_{u,x} = \int R_{u,x}\, dx \qquad (5)$$

with similar definitions for $L_{u,y}$ etc. The area under the correlation curves are definite even though the correlations approach zero asymptotically; the integral or macro scales of turbulence are thus precisely defined and represent average eddy sizes more appropriately. Integral time scales may be defined similarly in terms of the autocorrelation coefficient and an integral on dt instead of dx.

Another turbulence scale is often considered and needs to be defined. This is the <u>microscale</u>, so termed because it is indicative of the smaller eddy sizes present. It is also termed the <u>dissipation length</u> because it is intimately tied up with the rate of dissipation, i.e. the conversion of the turbulence energy into heat by viscous action.* Without attempting to prove these statements, it will be noted merely that the microscale also is determinate from the correlations. For small distances (δx close to zero) the correlation coefficient curves are approximately parabolic. The size of the parabola is measured by λ as indicated by the dashed line in Figure 3. A more precise definition of the microscale is given by the equation,

$$\frac{1}{\lambda^2} = 2 \lim_{\delta x \to 0} \frac{1 - R_{ux}}{\delta x^2} = \frac{1}{u'^2} \overline{\left(\frac{\partial u}{\partial x}\right)^2}$$

The second relation indicated for λ is convenient to use when only single-probe velocity measurements are made. In such a case, the mean temporal derivative of the turbulent velocity $\overline{\left(\frac{\partial u}{\partial t}\right)^2}$ is obtained with an electronic differentiating circuit. The derivative with respect to x is inferred from this with the aid of a relation (known as Taylor's hypothesis) concerning the transport of turbulence past the ob-

* It is often stated that λ represents the scale of the dissipating eddies. Townsend carefully points out that the scale of these eddies λ_d is considerably smaller. From his book (1956) (p. 46) it appears that λ_d/λ is approximately $3.2\, R_\lambda^{-1/2}$ where the turbulence Reynolds number $R_\lambda = \lambda u'/\nu$ may be high as 200 in shear flows.

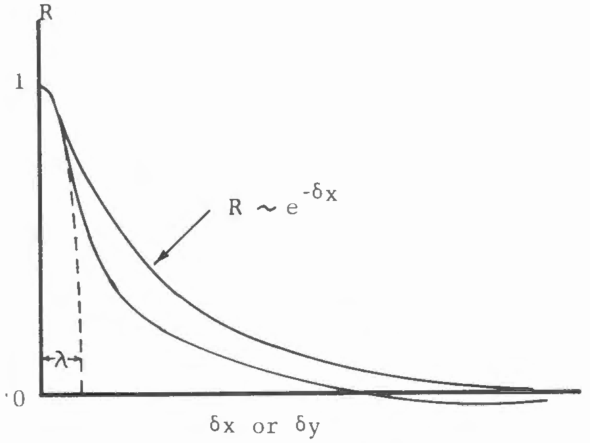

Figure 3. Typical Correlation Coefficient Variation

servation point. This assumes that,*

$$\frac{\partial}{\partial t} = -\bar{u}\frac{\partial}{\partial x} .$$

Thus, the definition of the microscale becomes:

$$\frac{1}{\lambda^2} = \frac{1}{\bar{u}^2 u'^2}\overline{\left(\frac{\partial u}{\partial t}\right)^2} \qquad (6)$$

Just as there are several L scales depending upon which velocity components are involved in the correlation, so are there several λ scales. Some of these can only be found from the correlations; however, due to the availability of electronic differentiating circuitry, $\lambda_{u,x}$ or $\lambda_{v,x}$ are more commonly determined quantities than the macroscales.

One final item, of basic import and consideration in regard to turbulence, is that indicating the manner in which the energy of the fluctuations is distributed amongst the various sized eddies, i.e. how it varies with scale. This results in consideration of the turbulence spectrum. Since one may consider the assemblage of eddies to be transported past the point of measurement (in the usual Eulerian system), the spectrum is most commonly evaluated in terms of frequency as in considerations of the spectrum of sound. This is in contrast to the more classical concept of light spectra associated with the wavelength. Actually turbulence spectra may be presented as functions of the frequency n, the wavelength $\Lambda = \bar{u}/n$ of motion (corresponding to the eddy size), or the wave number

$$k = \frac{2\pi n}{\bar{u}} = \frac{2\pi}{\Lambda} .$$

* Experimentally this has been found valid for the smaller eddies but not the larger.

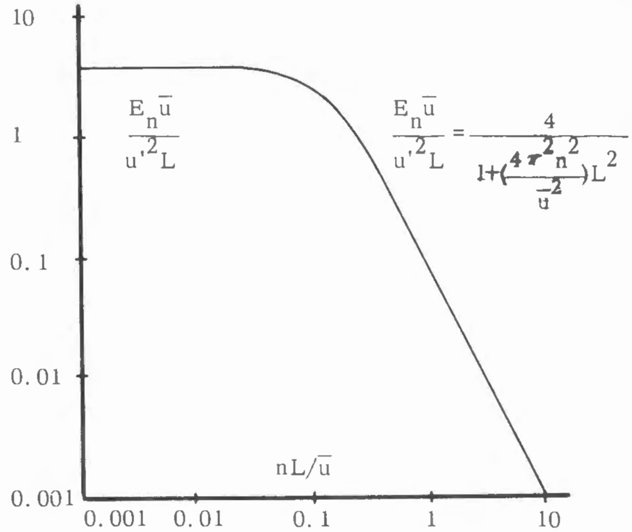

Figure 4. Turbulence Spectral Distribution.

The spectral function E_n (or E_Λ or E_k, similarly) is defined as the fraction* of the turbulent energy (divided by $\rho/2$) occurring between frequencies n and $n + \delta n$. The integral of this function over the frequency range is equal to the intensity squared,

$$u'^2 = \int_0^\infty E_n dn$$

Turbulence spectra from hot-wire-anemometer velocity-fluctuation measurements are found with the aid of wave or spectrum analyzers (essentially tunable band-pass filters) developed principally for acoustic purposes. In air the general range of frequencies of interest is the "hi-fi" region, or from 20 to 15,000 cycles per second. In water the frequencies are lower. Figure 4 indicates a typical distribution curve presented in dimensionless form as a function of a Strouhal number $nL/\bar{u}$ based on the macroscale of the turbulence. The curve drawn represents the trend of measured experimental data quite well.

The spectrum of turbulence is intimately related to the correlation functions previously introduced. In fact, G. I. Taylor in 1938 showed that each one is a Fourier transform of the other. Knowledge of the variation of one function may thus be determined from the variation in the other. Thus,

$$E_n = 4u'^2 \int_0^\infty R_t \cos 2\pi nt \, dt$$

* Actually $E_n \delta n$ is the energy between n and $n + \delta n$.

As Hinze* notes, only one of these functions needs to be determined to know all that they tell us about the nature of the turbulence, but actually neither is free from inaccuracies. Thus, "it is better to consider the relation between spectrum and correlation functions merely as welcome means of checking the experimental determination of both." Such verification has been indicated by several researchers. The equation indicated for the curve in Figure 4 is that derived from the correlation function curve approximated by the exponential function,** i.e., $R \sim e^{-\delta x/L}$. It is found to fit the experimental data quite well, thus checking the relation between spectrum and correlation.

Spectrum presentations of turbulence information permit one to consider how the energy of the turbulence is distributed among the various eddy sizes. Since the lower frequencies indicate the large eddies, or scales, and vice versa, inspection of Figure 4 indicates that much of the energy is associated with the large eddies (i.e., the macroscopic or integral-scale size). Above some frequency or below some size the energy is seen to drop off very rapidly with increase in frequency and decrease in size or scale. A flow of energy from one degree of freedom to another is indicated; in fact, from the large to the small eddies. The small scale components lose energy via the action of viscosity much faster than the large ones. The action of the non-linear inertia terms in the equations of motion have been shown to be responsible for the transfer of energy from the large scale motions, in which form it is extracted from the mean flow, to the small scale motions where it is dissipated.

The rate of turbulent energy dissipation has been shown to be simply related to the quantities already defined. In isotropic turbulence, the energy in the turbulence is

$$e' = \frac{\rho}{2}\left(u'^2 + v'^2 + w'^2\right) = \frac{3}{2}\rho u'^2$$

and the rate of dissipation per unit volume is found to be

$$\phi = \frac{de'}{dt} = 15\nu \overline{\left(\frac{\partial u}{\partial x}\right)^2} = 15\nu\left(\frac{u'}{\lambda}\right)^2 \qquad (7)$$

where ν is the fluid kinematic viscosity (ft.^{2}/sec.). In more general anisotropic turbulence, e' is represented only by the first expression above and the dissipation function ϕ involves six $\overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2}$ terms. However, the same type of relation -- as indicated above for the isotropic case -- may be inferred with the constant 15 replaced by an unspecified factor,

$$\phi = K\nu\left(\frac{u'}{\lambda_x}\right)^2 . \qquad (8)$$

The divergence of the value of K from the isotropic value of 15 is an indication of anistropic effects. Since the dissipation is associated with the fine scale motions and these are thought to be nearly isotropic, no great divergence is expected. Near the center of a pipe or channel K is about 15, if the intensity is taken as the average of the three components. Using u' only, Laufer in 1950 found K = 17 in this region. Nearer the wall and in other shear flow cases the value of K is higher. Thus in the flat-plate boundary layer Klebanoff in 1954 found an average value of 32 to apply. In homologous turbulence (plane Couette flow) preliminary measurements indicate values in the range 14 to 28, or an average value of around 20. The turbulent microscale is found to be more or less constant, at least in air, (about 0.01 feet) implying that the size of the eddies associated with dissipation is roughly a constant. The variation in turbulent energy dissipation over a flow field thus appears to be roughly proportional to its intensity.

IV. ORIGIN AND PRODUCTION OF TURBULENCE

Consideration of turbulence production gives valuable insight into its nature. From the discussions just concluded it appears that, roughly, the dissipation of turbulence is proportional to its intensity. The strong variation in turbulence intensity noted in shear flows must therefore originate in the production phases. Energy is continually extracted from the mean flow, as represented by $\bar{u}$. In a pipe or conduit this energy comes from the favorable pressure gradient; in simple flow past a wall or boundary it results in a thickening of the lower energy* region known as the boundary layer. For the simple case of the flat plate** boundary layer the rate of growth of a quantity, termed the energy thickness and representing the deficiency in flow energy, is given by the relation

$$\bar{u}_1^3 \frac{d\delta_e}{dx} = 2\int_0^{y>\delta} \left(-\overline{uv} + \nu\frac{d\bar{u}}{dy}\right)\frac{d\bar{u}}{dy}\, dy, \quad (9)$$

where $\bar{u}_1$ is the mean velocity outside the boundary layer, and the energy thickness is defined as

* J. O. Hinze, Turbulence. New York. McGraw-Hill Book Co., 1959, p. 60.

** See H. L. Dryden, Proceedings 5th International Congress of Applied Mechanics, 1938, p. 362.

* of kinetic energy less than in the free stream outside the layer . . .

** Conventionally a flat plate is not only a plane surface but also one along which fluid flows without pressure gradient.

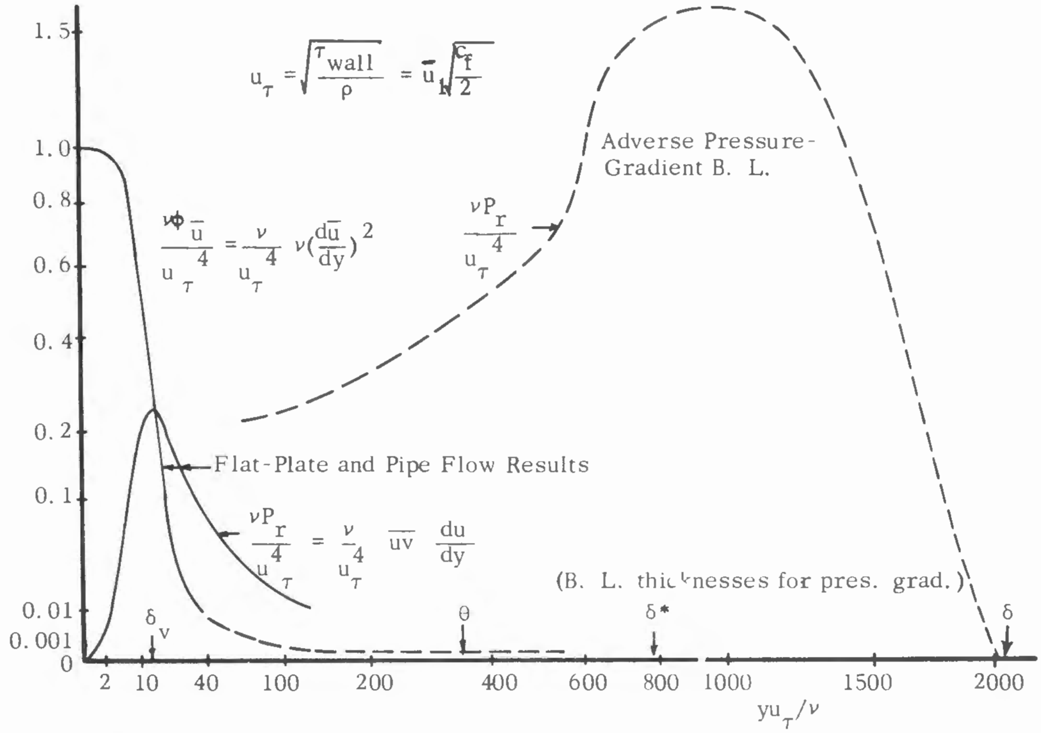

Figure 5. . Cross-stream Variation in Direct Dissipation and Turbulence Production Terms (Proceedings American Society of Civil Engineers, Vol. 83 (1958), paper 1393).

$$\delta_e = \int_o^{y>\delta}\left(1 - (\frac{\bar{u}}{\bar{U}_1})^2\right)\frac{\bar{u}}{\bar{U}_1}\,dy.$$

The above equation for the rate of extraction of mean-flow energy indicates that some of it goes directly into heat (the second term in equation 9) through the action of viscosity, but most of it goes into turbulence through the action of the Reynolds shear stress on the mean velocity gradient (i.e., as $\overline{uv}\,\frac{d\bar{u}}{dy}$). This turbulence is dissipated into heat as indicated by the relations already presented. As an integral equation only the average conditions across the flow are indicated. In detail there may be considerable flux of energy across the flow and between the several turbulence components. As indicated in Figure 5 for two boundary layer flows, the local direct dissipation of mean-flow energy into heat

$$\phi_{\bar{u}} = \nu(\frac{d\bar{u}}{dy})^2$$

is negligibly small except very near the bounding wall, where viscous forces predominate. In the flat-plate boundary layer case, as the distance y from the wall is increased the turbulence production (from the mean flow)

$$Pr = \overline{uv}\,\frac{d\bar{u}}{dy}$$

decreases slowly from a peak value near the wall. The peak rate of turbulence production appears at a wall distance $\delta_v \cong 12\nu/u_\tau$ known as the viscous film or sublayer* thickness. This represents the point at which the viscous forces or stresses are the same order of magnitude as the turbulent ones. Inside this very small distance, the viscous forces predominate over the turbulent ones, while outside the reverse is true. At first glance it is surprising that the greatest transfer of energy from the mean flow to turbulence in negligible pressure gradient flows (pipes, channels with fully developed flows and flat-plate boundary layer) occurs at the edge of this region. But this is the locale of the largest mean velocity gradient and turbulent shear, as will

* The appelation "laminar" sublayer used by most authors is inappropriate and "viscous" is to be preferred. The factor u_τ is $\sqrt{\tau_w/\rho}$ where τ_w is the wall shear stress.

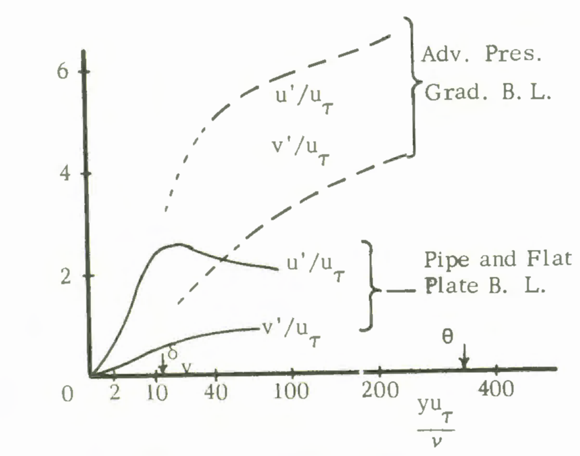

Figure 6. Turbulence Intensity Variations. (Proceedings of the American Society of Civil Engineers, Vol. 83 (1958), paper 1393)

appear in later sections of this presentation. In flows in which strong stream-wise variations in mean velocity and shear stress occur, as indicated by the adverse-pressure-gradient results, a second much higher peak in the turbulence production occurs relatively far from the wall. The very-near-wall conditions appear about the same, however, and the mean-flow dissipation is still negligible at most wall distances.

Detailed presentation of the history of the turbulence energy in shear flows is beyond the scope of this discussion. However, a brief summary of the significant occurrances will be attempted. It has already been noted that the turbulence dissipation is associated with the fine-scale motions (smallest eddies and highest frequencies), while its production is associated with the large scale motions, i.e., the large eddies. Further consideration of the equations indicating the turbulence production leads to the surprising conclusion that the turbulence is initially produced as a u fluctuation and only enters into the v and w fluctuations via pressure correlation terms ($\overline{up}$, etc.). Conceptually, the mechanism of this transfer toward isotropy, via pressure fluctuations, is as follows: the u fluctuations are produced by the action of the turbulent shear $\overline{uv}$ on the lateral mean-velocity gradient $d\overline{u}/dy$. Due to Bernoulli type relations, pressure fluctuations result from u fluctuations. These, similarly but in reverse fashion, produce v and w fluctuations. This helps explain the strongly anisotropic nature of the turbulence near the wall, where the production rate is quite high. As seen in Figure 6, for flat plate and pipe flows the value of u' or the r.m.s. value of the u fluctuation is found to peak at the edge of the viscous film, while v' is much less at this point and has no such peak. As the center of the conduit or the outer (away from the wall) region of the boundary layer is approached, the turbulence is found to approach isotropy.

Returning to the turbulence energy as an entity rather than as several components, there still remains the consideration of the history of this turbulence from the time and locale of its production to those of its eventual dissipation into heat via viscous action. As has been inferred earlier this is not necessarily a direct occurrence. In isotropic turbulence the production and dissipation are approximately zero, or if finite they occur at the same place. In homologous turbulence the production of turbulence energy at a point is balanced by its dissipation at that point. Temporaly of course, the turbulence must exist for a while so that this statement has only to do with the transverse position of the turbulence. The turbulence energy is produced at some wall distance y, floats down the stream and is dissipated into heat at this same wall distance. In nonhomogeneous, and hence anisotropic, shear flows the turbulence energy may be produced at one wall distance and dissipated at another.

Consideration of the pertinent equations characterizing the details of turbulence behavior indicates that between production and dissipation to heat the turbulence energy may be transported laterally by four different actions. These are:

a. convection (Townsend terms this advection) by the mean motion,
b. convection or diffusion by the turbulent motions,
c. transfer by work of the fluctuating pressure gradients,
d. diffusion or transport by viscous forces.

Since these are not all of like sense and are difficult of measurement (the pertinent pressure correlations for (c) have only been recently measured) - the general trends of the turbulent energy movement are not well defined. The transfer effects are significant and it appears that in the pipe and flat-plate boundary layer cases the turbulence energy undergoes an outward flux from the wall region. The turbulence level in the outer two-thirds of the flow is maintained by lateral transport (d) while in the inner (near wall) half of the flow there is the boundary-layer observation by Klebanoff (1954) that 85 percent of the total energy dissipation occurs very near the all (i.e. for $u_\tau y/\nu < 30$). A similar situation occurs in pipe flow.

V. TURBULENCE STRUCTURE IN SOME TYPICAL FLOWS

The variation in the turbulence structure behind grids in air tunnels and in two shear-flow cases will be briefly summarized at this point. In the first case the turbulence is isotropic so that all that is needed is information on the variation in intensity and scale with distance downstream of the grid. Typical experimental results are presented in Figures 7 and 8. The wake structure (vortex

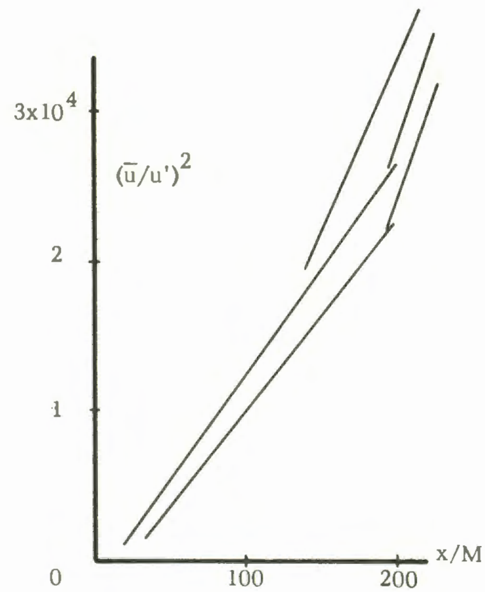

Figure 7. Isotropic Turbulence Decay (from Batchelor & Townsend, Proceedings of Royal Society of London, Vol. A193)

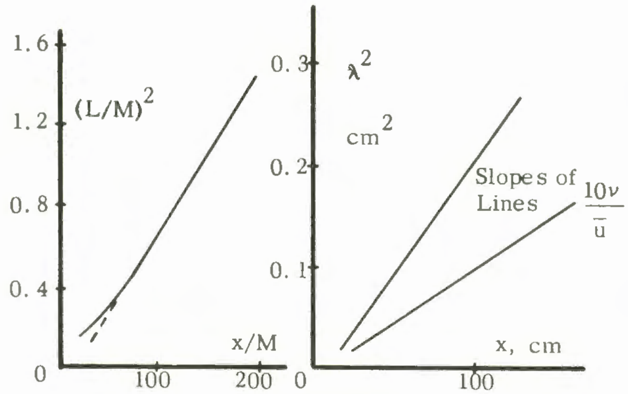

Figure 8. Scale Changes During Initial Period of Isotropic Turbulence Decay (from Batchelor and Townsend, Proceedings Royal Society of London, Vol. A193)

streets) behind the individual bars of the grid or mesh disappear and the isotropic structure is found after 15 or 20 times the mesh (bar spacing in grid) distance behind the grid. For about the next 200 mesh distances the intensity of the turbulence decreases inversely with the square root of the distance, or stated differently the turbulence energy e' decays and varies inversely with distance or time. The turbulence decay law is $(\frac{\bar{u}}{u'})^2 = a\frac{x-x_o}{M}$ where x_o represents the virtual origin of the turbulence and M the mesh of the grid producing the turbulence. The factor a depends upon the type of grid used; for biplane grids Batchelor and Townsend* found $a = 106/C_D$, while for single rows of cylinders about half this value, and for a single row of slats about 85 percent. The size of eddies increases since the integral or macroscale is seen to be proportional to the square root of the distance. As might be expected the large eddies are porportional to the mesh of the grid producing the turbulence. The microscale increases with the square root of the distance in accord with a relation specified by the kinematic viscosity and the mean velocity.

After some 200 mesh distances the character of grid turbulence begins to change from that noted for the so-called initial period. A final period occurs, after about 450 mesh lengths, in which the turbulent energy varies with the -5/2 power of time and distance rather than the -1 power as in the initial period. In the final period most of the energy resides in the large eddies, as might be inferred from the indicated increase in L with distance. One peculiar feature of the turbulence in the final period is that it appears strongly anisotropic. This presumably means that the large eddies, which did not decay rapidly, were produced by the grids in an anisotropic condition and remained essentially that way as the other, more isotropic, turbulence decayed. In the final period the microscale or dissipation length increases at a slower rate, i.e. λ^2 varies as $4\nu x/\bar{u}$ rather than $10\nu x/\bar{u}$ as in the initial period.

The most commonly studied shear-flow turbulence case is the fully developed conduit or flat-plate boundary-layer flow. Except for one interesting feature of the later flow the turbulence structures in these situations are the same.* They will therefore be discussed together following a note as to their differing feature.

For many decades fluid mechanicians treated boundary layer and conduit flows as being interchangeable. Thus in the early 1930's ** von Karman derived a logarithmic relation for the velocity profile and skin friction in a turbulent boundary layer growing along a flat plate, on the assumption that the flow in it was similar to pipe flow. The boundary layer thickness δ was assumed to be equivalent to the pipe radius $b = D/2$. The logic of this assumption is apparent in elementary presentations of the laminar-viscous boundary layer in which the

* Proceedings Royal Society of London, Vol. A193 (1948), p. 569, and Vol. A194 (1948), p. 527.

* See G. B. Schubauer, Journal of Applied Physics, Vol. 25 (1954), pp. 188-196.

** Even earlier (1920) Prandtl and von Karman applied the 1/7th power law velocity profile (observed in pipes) to the flat-plate boundary layer.

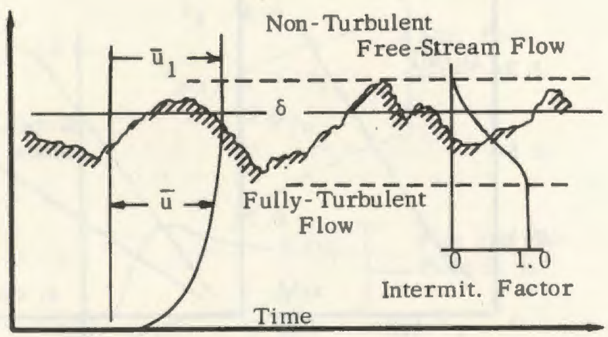

Figure 9. Intermittent Characterization of Outer Edge of Turbulent Boundary-Layer Flow

velocity profile is shown to be closely parabolic, exactly what it is in the pipe (i.e. Poiseuille) flow. Comparison with experimental observations of turbulent boundary-layer flows indicated discrepancies with the pipe assumption even for flat-plate flows. These were not explained until the intermittent nature of the flow in the bounding region between fully turbulent motion and the undisturbed fluid outside the layer was observed.

Intermittency was first reported by Corrsin (circa 1943) at the outer edges of jets. The flow in a bounding region of some width was seen to be turbulent part of the time and turbulence-free the rest of the time. Quantitative characterization is via the intermittancy factor, defined as the fraction of time that the flow is fully turbulent. Experimental indications in the boundary layer are that this is unity for y/δ values less than 0.4, reaches a value one-half at $y/\delta = 0.78$ or 0.80 and is almost zero at 1.2 times the boundary-layer thickness. The nature of the flow in this intermittent region as well as the variation in the intermittency factor is indicated in Figure 9.

Schubauer in 1954 showed that the turbulence characteristics of boundary-layer (flat plate) flows agreed with those found in pipes if the former were divided by the intermittency factor, i.e. when account is taken of the fraction of time that the outer region of the boundary layer is actually turbulent. As for the mean flow characteristics, the velocity profile and shear-stress variations differ between pipe or channel and boundary-layer flows. The situation for the shear stress is indicated in Figure 10. The differences in the away-from-the wall region are probably due to intermittency, those near the wall are associated with the small but sometimes significant pressure gradient which occurs in conduit flows. Differences in the mean velocity profile are also significant but not great. They are evidenced in the different constants applicable to the core (see Table 2).

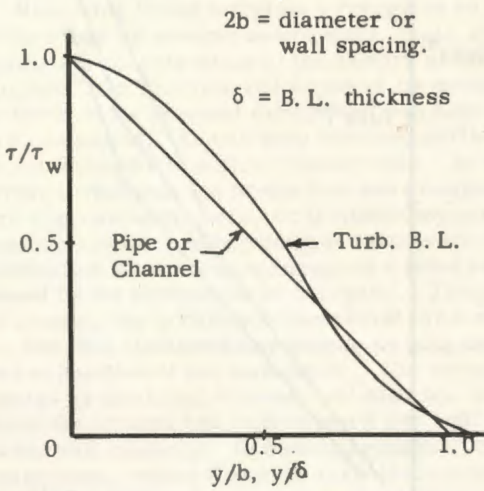

Figure 10. Shear Stress Variation Across Pipe and Boundary Layer

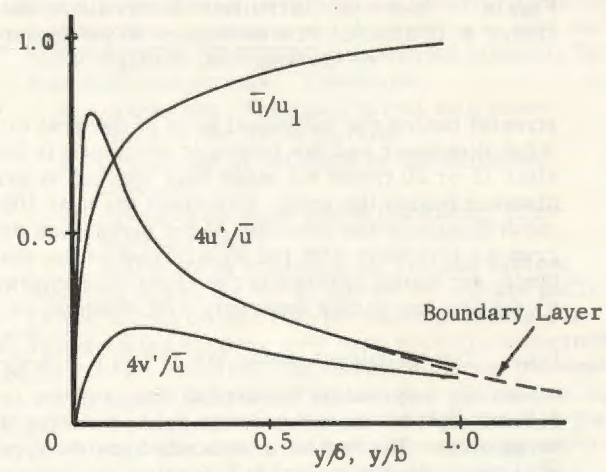

Figure 11. Mean Velocity and Turbulence Intensity

The general variation of turbulence intensities in turbulent boundary-layer, channel, or pipe flow is indicated in Figure 11. The third intensity component w', although not shown, reaches values midway between the other two, but with no near-the-wall peak. Determination of the turbulence scale in these flows has not been as extensively reported.* The more easily determined microscale λ_u, appears to be of the order of 0.01 feet in air, even for adverse pressure gradient flows (Robertson and Calehuff 1957), while in a channel Laufer (1951) found

* Somewhat more extensive scale data is available than has been reported herein. The general references (see end of this report) consulted do do not contain all of data and not all the original sources have been checked.

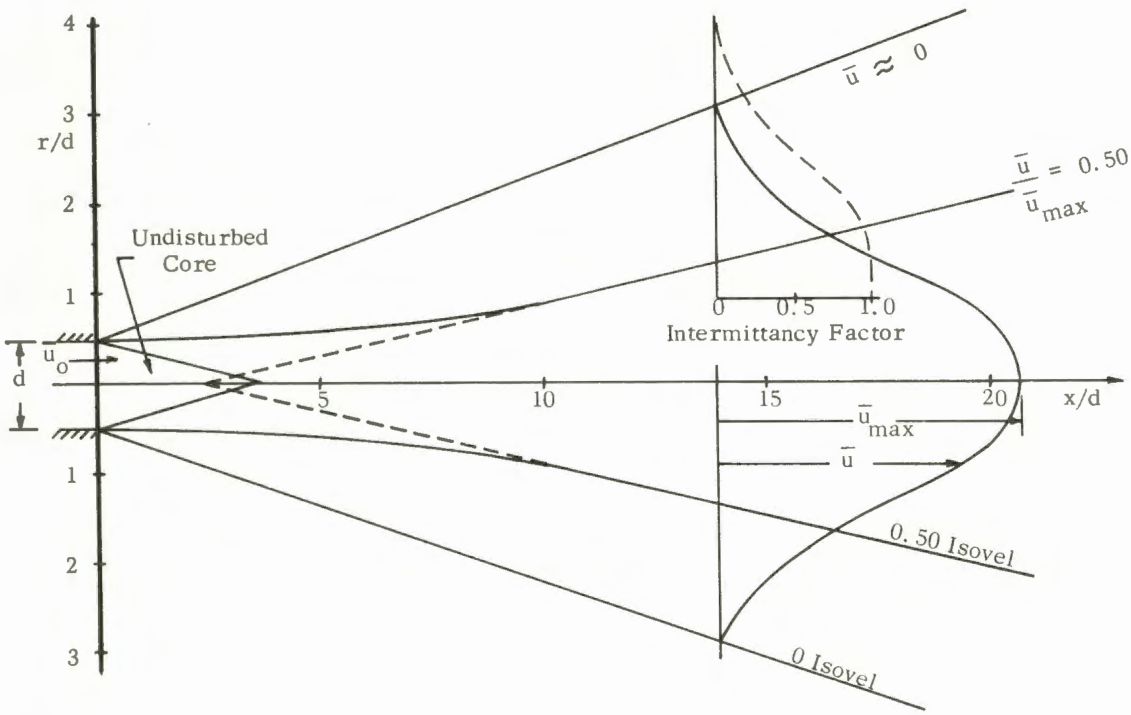

Figure 12. Jet Mean-Flow Pattern

it to vary from 0.01 ft. near the wall to 0.022 ft. near the center of the channel. The integral scale increases more definitely with distance from the wall. In the region away from the wall the lateral and longitudinal scales are not equal, indicating large elongated eddies of lateral dimension one quarter to one-half the pipe radius.

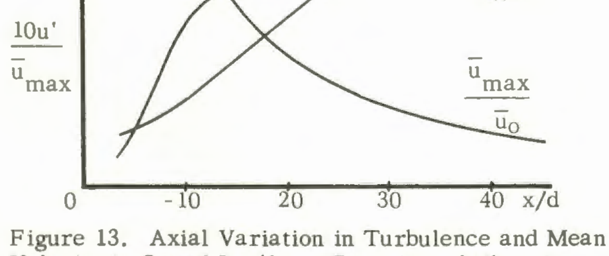

Figure 13. Axial Variation in Turbulence and Mean Velocity in Round Jet (from Corrsin and Uberoi, National Advisory Committee for Aeronautics, Technical Note 1865, 1949)

The second case of shear flow turbulence to be presented briefly is that associated with the intrusion of a jet into an otherwise still fluid. As diagrammed in Figure 12 the jet from a circular orifice or nozzle commences with a central core of undisturbed fluid bounded by turbulent mixing regions which spread and eat into it. These coalesce in some four or five diameters and, after a transition zone, a flow of similar velocity profiles occurs (after x/d of 10 to 20) in which the profile is closely expressed as the error function. The jet width increases linearly with distance from a virtual origin (at x = 0 to 10d), while the maximum velocity decreases with the inverse square of this distance. The longitudinal or axial variation in the significant quantities is indicated in Figure 13. The radial variation in turbulence is similar at various stations and is depicted in Figure 14 for the region of similar velocity profiles (x/d > 20). The intensities are seen to have peak values at the axis and drop off rapidly towards the edge of the jet.

For circular jet flows, the turbulence macroscale is found to be roughly constant in the transverse or radial direction and to increase linearly with axial distance ($L_x = 0.065x$), while the microscale is found to increase as the square root of the axial distance. The longitudinal macroscale is found to be about five times the transverse one, thus indicating eddies elongated in the flow direction, as has been noted for other shear flows. Just as in the

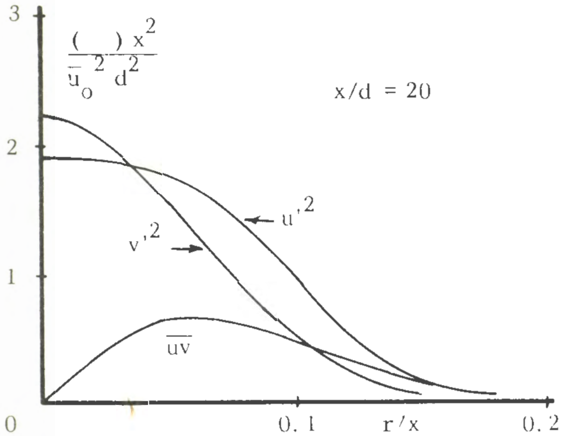

Figure 14. Radial Variation in Turbulence Shear and Intensity in Jet (from Corrsin, National Advisory Committee for Aeronautics, Report W94, 1943)

case of the outer regions of wakes and boundary layers, intermittency occurs along the sides of the jet as indicated in Figure 12; the intermittency factor has a value of about unity at the radial location where the velocity is half the maximum (0.50 isovel in Figure 12).

VI. TURBULENCE MODELS FOR MEAN-FLOW PROBLEMS

It should be apparent from the preceeding discussions and presentations, that there is still much we don't know about the nature of turbulent flows. Many technological situations occur in which solutions are required for mean-flow problems -- the velocity distributions and head loss or frictional relations being the usual desiderata. The statistical theory of turbulence, although yielding insight into the nature of turbulent motions, is unable to give such mean-flow solutions. Pragmatically one must therefore resort entirely to empiricism or develop some simple yet workable models of turbulence which do permit mean-flow solutions. Several such models have been introduced -- Boussinesq's eddy viscosity (1877), Prandtl's mixing length and momentum transfer (1924), Taylor's vorticity transfer (1932), and von Karman's similarity (1930, 1937) -- with varying degrees of suitability to the problem at hand as well as others not originally contemplated.

As in the case of phenomenological or behavioristic models used to aid the description and analysis of other physical problems, these turbulence models are not completely realistic in describing the occurrences. They are simplified models and the behavior assumed is not really measurable in the flow. For this reason the turbulence experts look askance at these and it is true that if used too religiously or in unusual situations they may be misleading. One feature of the model approach is that, at best, only the form of the mean-flow solution is obtained and one or more coefficients always remain to be obtained empirically via comparison with experimental results.

Perhaps the most usable phenomenological model for turbulence is the momentum-transfer, mixing-length model of L. Prandtl. This will be the only one presented here, since it has led to many practical solutions and is relatively easily understood. Consider a flow in the x direction with a lateral velocity gradient $\bar{u}$ = func (y). A fluid element or lump, considered for the moment to retain its entity without diffusing, has a mean velocity $\bar{u}_1$ at point 1 in Figure 15. At some instant a lateral turbulence fluctuation $\underset{\sim}{v}$ appears to transport this lump laterally some distance. It is assumed that the lump moves a lateral distance ℓ -- the mixing length -- before it loses its identity and acquires that of the fluid at point 2. Considering the new point, where the lump is to stop and mingle with the ambient fluid, the mean velocity there is $\bar{u}_2 = \bar{u}_1 + \ell \frac{d\bar{u}}{dy}$. The lump, however, arrives at point 2 with the instantaneous x-component velocity that it started with, i.e. $\bar{u}_1$. Its arrival hence introduces a longitudinal turbulence fluctuation.

$$\underset{\sim}{u} = -\ell \frac{d\bar{u}}{dy}$$

The longitudinal and lateral turbulence velocity components $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are thus negatively correlated in a lateral velocity gradient (cf. Figure 2). This discontinuous model of turbulence fluctuations is analogous to the molecular motions considered in the kinetic theory of gases, with the mixing length analogous to the mean free path. Although the real fluctuation process is continuous, this model is still usable. It leads to an understanding of the process by which turbulence in a mean-velocity-gradient

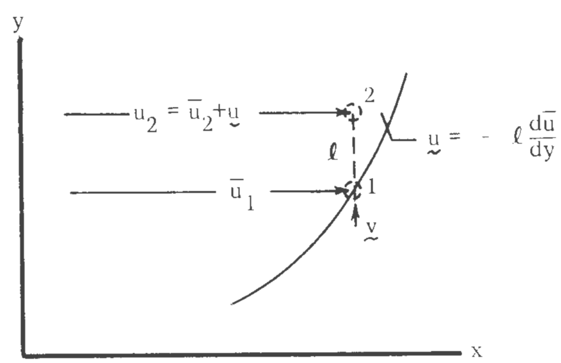

Figure 15. Turbulence Mixing Length and Velocity Fluctuations

Table 2

Mean-Flow Solutions Via Mixing-Length Model

Mixing Length \ Shear Stress	Constant $\tau = \tau_w$	Linear $\tau \sim (b-y)$
Linear $\ell \sim y$	Wall region of bound. layer and pipe. $\frac{\bar{u}}{u_\tau} = 5.6(1 + \log \frac{yu_\tau}{\nu})$ (11) (Prandtl 1925, 1930)	
Constant $\ell = L$	Core region of plane Couette flow. $1 - \frac{\bar{u}}{\bar{u}_1} = 4.1\sqrt{\frac{C_f}{2}}(1 - \frac{y}{b})$ (13) (Robertson, 1959)	Core region of pipes, channels and bound. layer. $1 - \frac{\bar{u}}{\bar{u}_1} = D(1 - \frac{y}{\delta})^{1.5}$ (12) Pipe and Channel $D = 5.1\sqrt{C_f/2}$ F.P. Bound. Layer $D = 8.4\sqrt{C_f/2}$ A.P.G. Bound. Layer $D = 0.3$ to 1.3 (Prandtl 1925; Ross 1953)

τ_w = wall shear stress, $\bar{u}_1 = \bar{u}_{max.}$ or velocity midway between walls (Couette flow).

2b = wall spacing; for core law in pipes and channels replace δ by b.

$u_\tau = \sqrt{\tau_{w/\rho}} = \bar{u}_1\sqrt{C_f/2}$

flow produces shear stresses, via lateral momentum transfer, just as in the kinetic theory of gases the similar model leads to an explanation for viscous shear stresses resulting from molecular diffusion.

The Reynolds stress indicates the relation between shear stress and the turbulence fluctuations

$$\tau'_{yx} = -\rho\,\overline{uv}$$

Assuming $\underset{\sim}{v}$ to be proportional to $\underset{\sim}{u}$ and absorbing the proportionality factor into ℓ, one may write

$$\tau_{yx} = \ell^2 \left(\frac{d\bar{u}}{dy}\right)^2 \quad \text{or} \quad \ell^2 \left|\frac{du}{dy}\right| \left(\frac{du}{dy}\right). \quad (10)$$

All that is now required to obtain the mean-velocity solution $\bar{u}$ = func (y) is the variation in shear stress and mixing length. This may seem like quite a lot remaining yet to be specified, but actually considerable progress towards the solution has been made. The shear stress variation in the lateral direction, i.e. with y, is relatively simple (cf. Fig. 10) in many flows, while simple relations for the mixing length variation may be postulated on a rational basis. Prandtl's equation may thus be integrated to yield the desired relation, at least in certain cases.

Solution for the mean-velocity profile have been found by integration in three cases which are applicable to a wide variety of practical flow problems. Inspection of the shear stress curves of Figure 10 indicates a considerable region -- the core or away-from-the-wall region -- in which the shear stress varies linearly with distance from the wall. For adverse-pressure-gradient boundary layers such a linear shear-stress variation has been shown to be

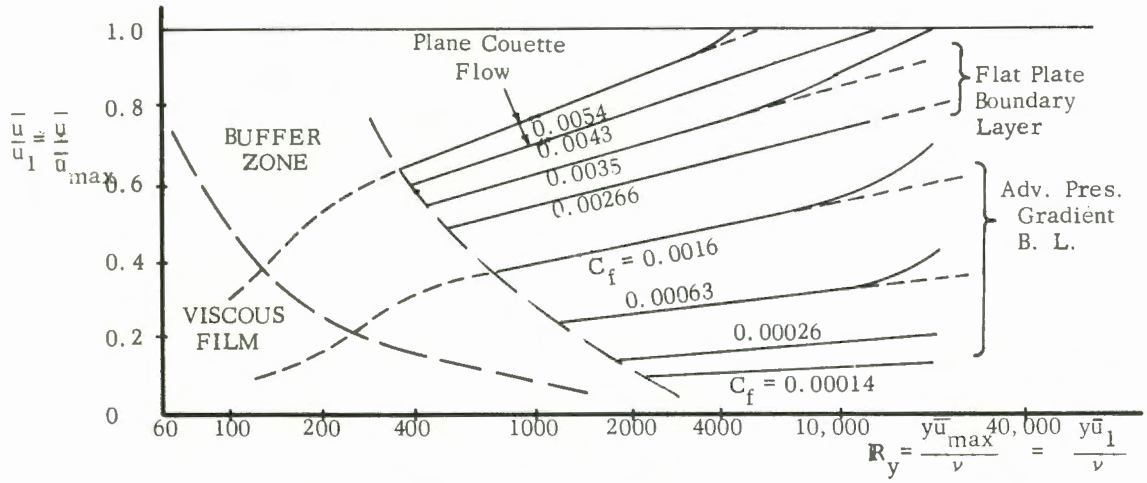

Figure 16. Wall-Law Velocity Variations

characteristic of the core region.* On the other hand, near the wall in the flat-plate boundary layer the shear stress is essentially a constant while in pipe, channel, and other boundary-layer flows it does not vary greatly in the near-wall region. Therefore from the shear stress standpoint, two distinct regions are noted -- a wall region with effectively-constant shear stress, and a more extensive core region of linear shear-stress variation. The situation is less specific for the mixing length, since this is hardly a directly measurable quantity. However the same two regions are identified. In the wall region, it is logical to assume that the size of the eddies (i.e. the mixing length) increases directly with y, the distance from the wall, and there is some evidence to justify this assumption. In the core region, the mixing length seems to be roughly a constant as for the macroscale in jet flows. The assumptions relevant to the solution of Prandtl's equation in these two regions are indicated in Table 2, with mention of the actual flows in which such assumptions apply. One additional case is included, that is which the shear stress and mixing length are both constant. As noted this is appropriate to the core region of plane Couette flow.

The mean-velocity profile distributions predicted from these mixing length analyses are indicated in this table. The first of these, currently termed the "law of the wall," has been well verified and is rather well accepted. It appears to be applicable in most flows, no matter what the outer or core flow is doing, for a region extending out from the wall some ten or fifteen percent of the boundary-layer thickness or pipe radius. Figure 16 indicates the near-wall velocity variations found for several kinds of flows.* Right near the wall, this logarithmic profile relation is replaced by a linear one in the thin film where the viscous forces predominate over the turbulent ones. This viscous film, more usually termed the laminar sublayer although this is not quite the right name for it, has thickness δ_v of about $11.6\nu/u_\tau$ as has been indicated in Figures 5 and 6.

The solution for the core region of pipe, channel, and boundary-layer flows is not as well accepted but does appear to yield a useful and technically adequate indication of the velocity variation. The predictions for the velocity profile in adverse pressure gradient flows is shown in Figure 17. The wall and core solutions also form a suitable basis for correlating of the friction and wall shear-stresses in these turbulent flows.

Prediction of mean velocity profiles for jet flows may also be based on the mixing-length momentum-transfer model. Thus, following Prandtl, Tollmien in 1926 assumed that the mixing length is constant radially (as is the integral scale) and is proportional to the width of the mixing zone; and hence it varies linearly with axial distance. The solution is not as straightforward as those outlined above.

* cf. Journal of Applied Physics, Vol. 21 (1950), pp. 557-561.

* This figure represents merely a verification of the form of the wall law. The skin-friction coefficients C_f are very difficult of direct measurement. As for the flows shown, C_f is best found indirectly by the closest fit between the experimental data and the analytical expression for the wall law. This represents a reliable useful technique for experimental determination of C_f.

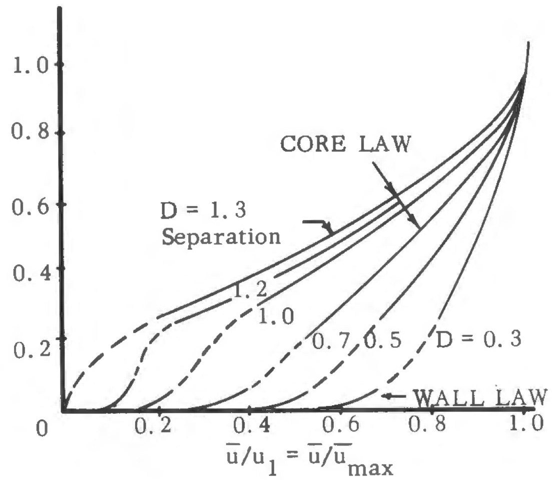

Figure 17. Adverse Pressure Gradient Boundary Layer Velocity Profiles (from Transactions American Society of Mechanical Engineers, Vol. 82, p. 209)

VII. SOME VELOCITY PROFILE CHARACTERIZATIONS

In analyzing turbulent flows certain mean flow parameters characterizing the velocity profile are of interest and information on these is not always available. This section of the turbulence discussion attempts to fill this void for simple flows.

The temporal-mean velocity-profile solutions just indicated may be used as the basis for predictions of the variations of pertinent factors in turbulent flow. Thus, for conduit and flat plate boundary-layer flows, in the early 1930's von Karman's "similarity" and Prandtl's "mixing length" theories were taken to predict a "universal" (i. e., extending clear across the flow from δ_y to δ or b) velocity-profile relation which still appears in many texts. For the last decade, however, evidence of the need for separate consideration of the wall and away-from-the-wall regions (as indicated by the separate solutions in the last section, Equations 11 and 12 of Table 2) has been apparent. The universal results were valuable in permitting prediction of expected relations between important mean-flow factors, although adjustment in constants was necessary to permit agreement with experiments. Thus, the relation between pipe friction factor f and pipe Reynolds number $\mathbb{R}_d$ in smooth pipes is

$$\frac{1}{\sqrt{f}} = 2 \log (\mathbb{R}_d \sqrt{f}) - 0.8 \tag{14}$$

Turbulent boundary-layer predictions on the assumption that δ correspond to $b = d/2$ were also reasonably successful for the skin-friction coefficient but with differently adjusted constants.

As implied by the preceding discussions the detailed fit of the "universal" profile to pipe flows is not good while its fit to flat-plate boundary-layer flows is much poorer, as a result of the influence of intermittency. As a technically useful characterization of the velocity profile we have the shape parameter

$$H = \frac{\delta^*}{\theta} = \frac{\int_o^{\delta} (1 - \frac{\bar{u}}{u_1}) dy}{\int_o^{\delta} (1 - \frac{\bar{u}}{u_1}) \frac{\bar{u}}{u_1} dy} \tag{15}$$

On the basis of the universal law one finds

$H = \dfrac{1}{1 - 3.5 \sqrt{c_f}}$ but, as shown in Figure 18, this yields too small a result.* For correlating boundary-layer flows a Reynolds number $\mathbb{R}_\theta$ based on the momentum thickness θ (an integral measure of the boundary-layer thickness, defined as the denominator in Equation 15) is found to be most appropriate. In fact, use of this parameter seems to circumvent many of the differences between pipe, two-dimensional channel, and flat-plate boundary-layer flows. Figure 18** thus indicates the observed variation in the shape parameter H with various flows of these types. Although there is some scatter, $\mathbb{R}_\theta$ is seen to be a suitable scaling parameter for both two-dimensional flows: those in which the pressure gradient is zero (flat-plate boundary layer) and those in which it is finite but has negligible influence on the profile (fully-developed conduit flows).

Two other velocity-profile shape parameters

---

* An even poorer indication is that of the power law $\bar{u} = Cy^{1/7}$ which was used as the first estimate of turbulent boundary-layer flows in extension of its more successful employment in pipes.

** This figure is similar to one presented by D. Ross in 1953, but includes much more data from various flows. The assistance of Mr. J. D. Martin in these calculations is gratefully acknowledged. Mr. M. Stevenson obligingly sent the author numerical data on his traverses. Data sources are: J. Laufer, Journal Aeronautical Science, 1950 and NACA Report 1053; F. Page, W. H. Corcoran, W. G. Schlinger, and B. H. Sage, Industrial and Engineering Chemistry, Vol. 44 (1952); M. Stevenson, University of Maryland, Institute for Fluid Dynamics and Applied Mathematics, Technical Note BN-147 (1958).

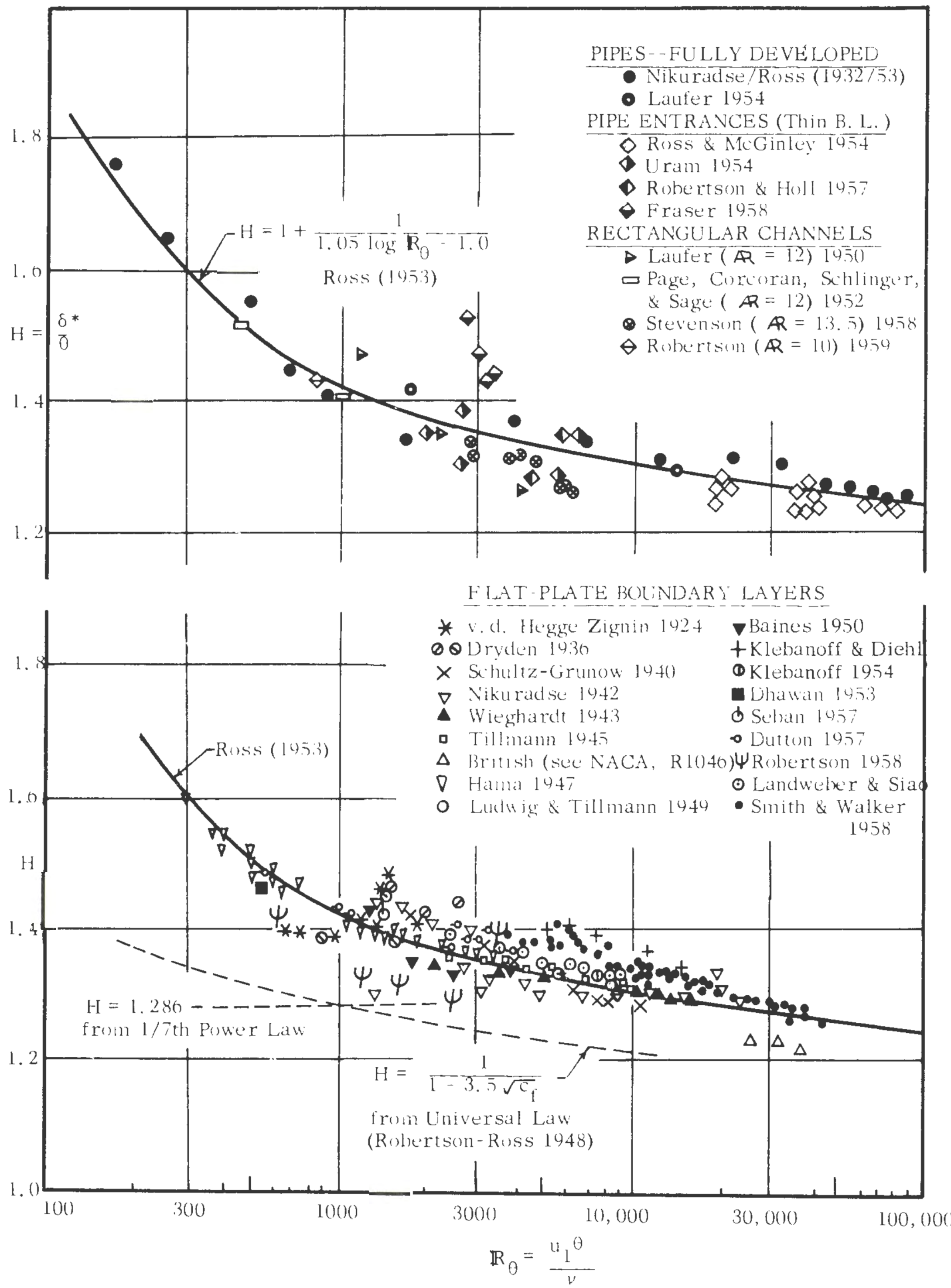

Figure 18. Shape Parameter for Flow Past Smooth Surfaces in Flat-Plate Boundary Layer Flow and Conduits.

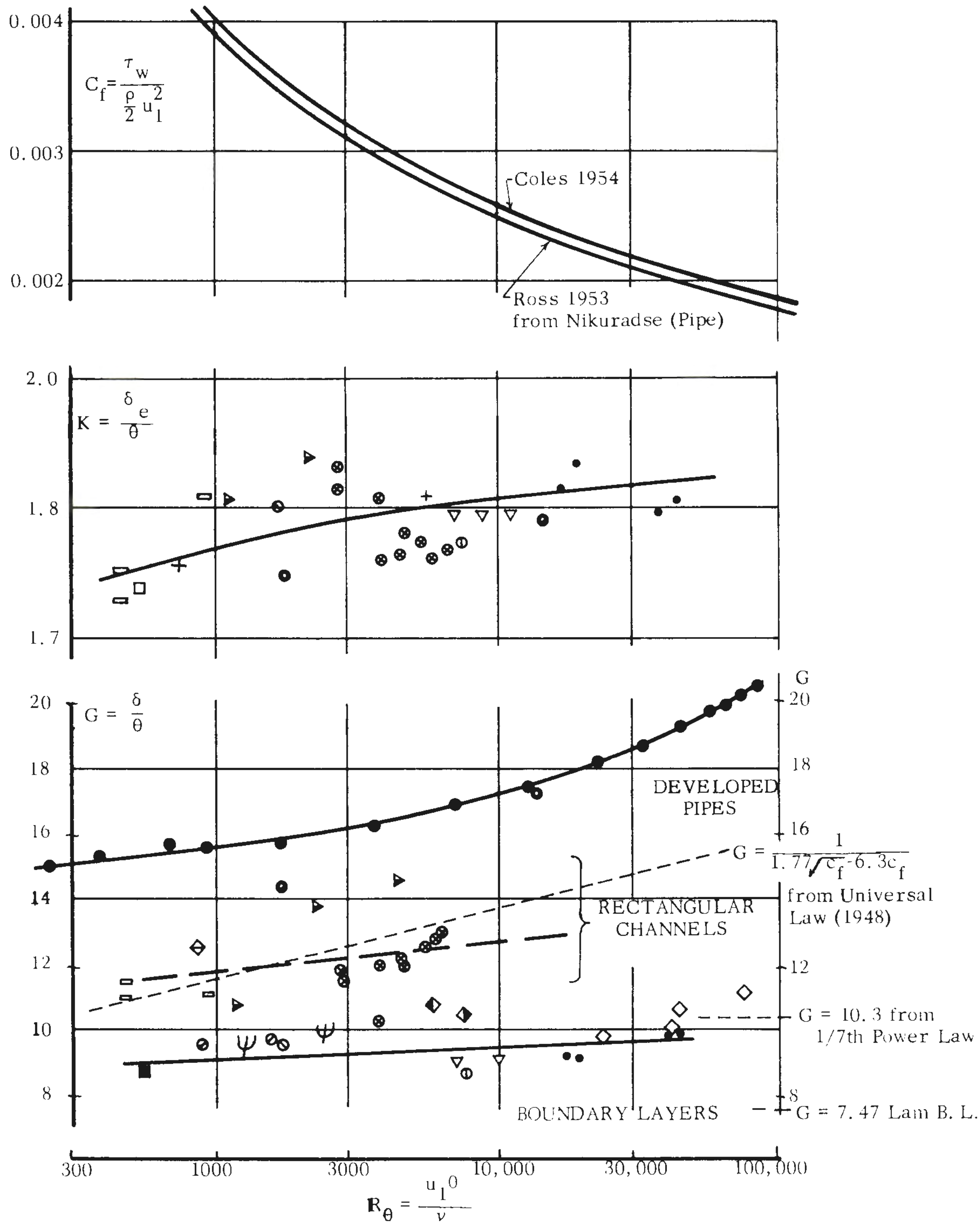

Figure 19. Turbulent Flow Parameters in Flat-Plate Boundary-Layer and Conduit Flow Past Smooth Surfaces.

are encountered in turbulent flow analyses and are hence recorded here. These involve one more integral thickness -- the energy thickness δ_e already introduced -- and the apparent (or disturbance) thickness δ. This latter is conventionally defined as the lateral y-distance where $\bar{u}/\bar{u}_1$ reaches a value of 0.99. For fully-developed pipe and channel flows it is merely half the diameter or wall spacing. These two thicknesses are taken in ratio to the momentum thickness to define the final two parameters,

$$G = \frac{\delta}{\theta}, \qquad K = \frac{\delta_e}{\theta}$$

Evaluation of G and K has been less common than H but the limited information which the author has been able to collect or calculate from available traverses is presented in Figure 19. The rectangular channel data presented here and in Figure 18 is based on data from numerous "two-dimensional" channel tests. The channel data on H and K are seen to agree well with the pipe and boundary-layer data. Significant differences appear for G between the several flows, as might be expected. The universal law prediction for G is again seen to be poor, except possibly for channel flows.

Also included in Figure 19, to complete this assemblage of flow characterization information, is a plot of the local skin-friction coefficient C_f as a function of $\mathbb{R}_\theta$ from two rather well-accepted sources. The line by D. Ross is based on a careful reanalysis* of J. Nikuradse's thorough study of fully-developed flow in smooth pipes. Ross also showed this line to agree well with flat-plate boundary-layer results, as is apparent from its proximity to the line established by D. Coles from boundary-layer data.

When flows, such as discussed in this section, are subjected to adverse (rising in flow direction) pressure gradients the friction and profile shapes are changed. A turbulence primer being hardly the place to delve into such complexities, we merely note that H increases while C_f and G decrease. Adverse pressure-gradient flows, if carried far enough, usually terminate with separation of the flow from the boundary; at such a point H may reach a value of 3 or 3.5 and C_f becomes zero. One method of analyzing such flow problems involves the D parameter of Equation 12 (away-from-the-wall law) and this parameter increases from normal values of about 0.3 to about 1.3 at separation. The general nature of the associated changes which occur in the velocity profiles are indicated in Figures 16 and 17.

* Proceedings of the Third Midwest Conference on Fluid Dynamics, University of Minnesota, 1953.

VIII. TURBULENCE AS A TRANSPORT MECHANISM

Just as in laminar-viscous motions, where the molecular activity not only makes itself felt as the viscosity in producing a shear but also as a mechanism to transfer (diffuse) heat, mass, and energy, turbulence similarly acts as a transport mechanism. Thus in many processes, turbulence acquires a significance beyond that of merely specifying the velocity profile and friction relations.

Analogous to the molecular transfer relations, which may be termed Newton's, Fourier's, and Fick's laws, we have the following relations for the turbulent transfer of momentum, heat and mass:

$$\tau_{yx} = -N \frac{d}{dy} (\rho \bar{u})$$

$$q_y = -K \frac{d}{dy} (\bar{T})$$

$$c_y = -\mathcal{D} \frac{d}{dy} (\bar{C})$$

In these relations q_y is the rate of heat flow in the y direction, T is the temperature, c_y is the time rate of diffusion in the y direction and C is the mass concentration per unit volume (as the number of some particular molecules or particles). Here N is the (turbulent) eddy viscosity -- or $v\ell$ from the mixing-length model - and K and $\mathcal{D}$ are the turbulent (or eddy) thermal conductivity and diffusion coefficient. The mixing-length model suggests that these eddy-transport coefficients are simply related by the dimensionless parameters

$$\mathbb{S}_{turb} = \frac{N}{\mathcal{D}} = 1 \qquad \mathbb{P}_{turb} = \frac{C_p N}{\rho K} = 1$$

where $\mathbb{S}$ and $\mathbb{P}$ are the Schmidt and Prandtl numbers and C_p is the fluid specific heat. Somewhat limited experiments have indicated that for particles in water $\mathbb{S}_{turb} = 1$ and for gasses in air streams it is about 0.6, while for heat transfer in air $\mathbb{P}_{turb}$ is 0.7 to 0.8. Taylor's vorticity transfer theory has resulted in somewhat better heat transfer correlations.

IX. DIFFUSION IN TURBULENT FLOWS

One final facet of turbulent flow discussed in this primer indicates an application of basic turbulence theory to a problem of some practical import. This is in the matter of diffusion of material in turbulently flowing fluids. Not only do such problems arise in many areas but turbulent diffusion has also

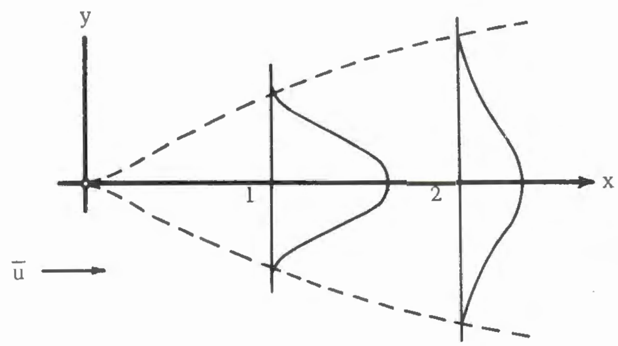

Figure 20. Lateral Spread of Materials.

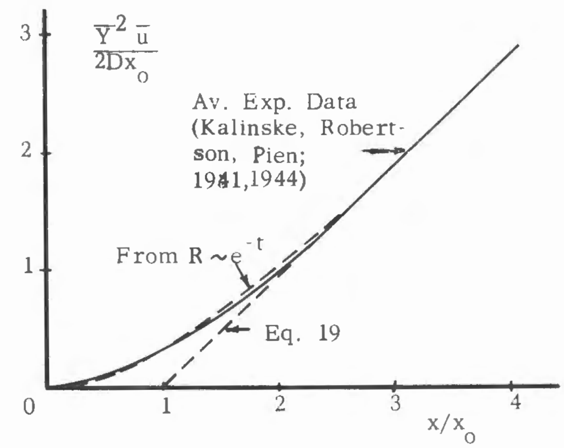

Figure 21. Generalized Plot of Diffusion Data Compared with Theory.

been employed as technique for studying the structure of turbulence* in shear flows. The basic nature of turbulence diffusion in theory and experience is briefly elucidated.

When some observable, or tagged, particles or fluid elements are released from a point in a turbulent stream, they will spread or diffuse laterally due to the action of turbulence as they are carried along. Observation of the passage of a considerable ensemble of such particles at downstream stations yield distributions such as indicated in Figure 20. Experimentally these may be found as histograms, or bar graphs based on finite increments in y, but the general appearance is the same and usually the distributions satisfactorily approximate the Gaussian relation whose form (for $\underline{u}$) is indicated in Figure 1. The standard deviation σ thus characterizes the spread or diffusion. In terms of a mean flow in the x direction (see Figure 20) with y the lateral coordinate, the variance (σ^2) of each distribution is calculated as

$$\overline{Y^2} = \frac{1}{N} \Sigma(y - \bar{y}_o)^2$$

for a large number N of particles observed. Here $\bar{y}_o$ is the mean y coordinate of the distribution and is zero if there are no bouyancy effects while x is in the direction of the mean velocity.

* cf. G. B. Schubauer, "A Turbulence Indicator Utilizing the Diffusion of Heat," NACA Report 524 (1935); T.K. Sherwood, W. L. Towle, R.B. Woertz, and L.A. Sedar, three papers on eddy diffusion in Industrial and Engineering Chemistry, Vol. 31 (April, August 1939); A.A. Kaliuske and J.M. Robertson, "Turbulence in Open Channel Flow," Engineering News Record, April 1941; A.A. Kalinske and C. L. Pien, "Eddy Diffusion", Industrial and Engineering Chemistry, Vol. 36, (March 1944).

The observed variation in $\overline{Y^2}$ with x is shown by a solid line in Figure 21. This represents the trend line from a considerable number of experiments and is made dimensionless with the aid of two parameters to be identified.

The observations of diffusion of particles from a point source in a turbulent stream are in general agreement with a 1921 theory, due to G. I. Taylor, termed "diffusion by continuous movements." Based on the Lagrangian autocorrelation coefficient (Equation 3) the mean particle spread rate is

$$\frac{d\overline{Y^2}}{dt} = 2\,\overline{v'^2} \int_o^t R_{Lt}\,dt \qquad (16)$$

The exact form of the correlation coefficient is not known, but as noted in an earlier section, it is not greatly different from e^{-t}. The relation (Equation 16) for the temporal rate of lateral diffusion is changed to a spatial variation with the aid of Taylor's hypothesis* so that t is replaced by $x/\bar{u}$. The nature of the diffusion is easily found for two limiting regions. At small distances (small times) R is obviously unity and Equation 16 becomes

$$\frac{d\overline{Y^2}}{dt} = \frac{2\overline{v'^2}x}{\bar{u}^2} \qquad (17)$$

for small x. As a means of determining turbulence structure, Equation 17 permits an evaluation of v' from measured variations in $\overline{Y^2}$ with x in a known $\bar{u}$ flow. This technique was first employed by G. B. Schubauer with heat diffusion (reference given earlier in this section). As the distance (or time) becomes large R reaches a zero value and the

* cf. discussion preceding Equation 6.

integral of Equation 16 is a constant

$$\int_0^\infty R\, dx = x_o \tag{18}$$

The diffusion equation then appears in the form

$$\frac{d\overline{Y^2}}{dx} = \frac{2\overline{v'^2}x_o}{\overline{u}^2}$$

for large x. Hence

$$\overline{Y^2} = \frac{2\overline{v'^2}x_o}{\overline{u}^2}(x - x_o)$$

The diffusion coefficient $\mathcal{D}$ of the previous section may be defined in terms of the slope of the $\overline{Y^2}$ curve at large distances,

$$\mathcal{D} = \frac{\overline{u}}{2}\left(\frac{d\overline{Y^2}}{dx}\right)_{\text{large } x} = \frac{\overline{v'^2}x_o}{\overline{u}}$$

The diffusion equation for large x is now written in normalized form as

$$\frac{\overline{Y^2}\,\overline{u}}{2\mathcal{D}x_o} = \left(\frac{x}{x_o} - 1\right) \tag{19}$$

Reference to Figure 21 indicates that after distances of about $2x_o$ the observed diffusions agree well with this prediction. Comparison of Equations 5 and 18 indicates that x_o is an integral length scale of turbulence, hence one can evaluate L from experimental diffusion studies.

The divergences between experiments for intermediate distances (less than $2x_o$) and theory when R is assumed to follow a negative exponential relation are seen to be not great (Figure 21).

It is significant that this is the same function for R as employed for predicting the spectrum in Figure 4. Very often the results for large distances are of prime importance in predictions of the turbulent diffusion of contaminant material in a fluid and only $\mathcal{D}$ is required. Of course, turbulent diffusion may be distorted by wall or boundary proximities or buoyant (gravitational) forces on the particles.

X. CLOSURE

The reader should now be in a position to peruse the literature of turbulence with understanding, and he will be better able to consider the import and effects of turbulence in the fluid-dynamics problems which he encounters. The true connotation of the term turbulence has been exposed.

Obviously, in a primer, complete development of all the complexities is hardly possible. Furthermore, as study of the following, and much more complete, general references will demonstrate, our understanding of turbulence is as yet too incomplete to permit such development even in the most advanced books.

XI. GENERAL REFERENCES

A. A. Townsend. The Structure of Turbulent Shear Flow. London, G. B.; Cambridge University Press, 1956.

J. O. Hinze. Turbulence. New York; McGraw-Hill Book Co., 1959.

C. C. Lin (Ed.) Turbulent Flows and Heat Transfer (Vol. V of High Speed Aero-Dynamics and Jet Propulsion). Princeton, N. J.; Princeton University Press, 1959. (Note: paperback versions of two portions -- 9. Turbulent Flow, and 10. Statistical Theories of Turbulence -- are also available from this publisher.)

PUBLICATIONS OF THE COLLEGE OF ENGINEERING

Bulletins from the University of Illinois College of Engineering are detailed reports of research results, seminar proceedings, and literature searches. They are carefully reviewed before publication by authorities in the field to which the material pertains, and they are distributed to major engineering libraries throughout the world. They are available at a charge approximately equal to the cost of production.

The annual *Summary of Engineering Research* is available in the fall of each year. It contains a short report on every research project conducted in the College during the past fiscal year, including the names of the researchers and the publications that have resulted from their work. It is available free of charge.

Engineering Outlook, the College's monthly newsletter, contains short articles about current happenings, new research results, recent technical publications, and educational practices in the College of Engineering. Free subscriptions are available upon request.

The Seminar and Discussion Calendar, which is published and distributed weekly, lists current meetings, lectures, and other events on the engineering campus that are open to the public. Free subscriptions are available upon request.

Requests for a catalog of available technical bulletins or for any of the above publications should be addressed to the Engineering Publications Office, College of Engineering, University of Illinois, Urbana, Illinois 61803.